On the first set $J_n(\alpha, \beta, K; x)$ of Bi-orthogonal Polynomials suggested by the Jacobi Polynomials

N. M. Kavathekar1*, G.A.Dhanorkar [2], Vidhyadhar Nalawade [3], T.B.Jagtap [4]

1 Department of Mathematics, Mudhoji College, Phaltan, Dist-Satara (M.S.) India.
2 Department of Mathematics, K.B.P. College Vashi, Navi Mumbai, India.
3 S. G. R. G. Shinde Mahavidyalaya, Paranda, India
4 Department of Mathematics, Y. C. Institute of Science, Satara (MS) India

Abstract

Madhekar and Thakare succeeded in constructing a pair of bi-orthogonal polynomials $J_n(\alpha, \beta, k; x)$ and $k_n(\alpha, \beta, k; x)$ that are suggested by Jacobi polynomials. In the sense that for $k = 1$ both these polynomials reduces to Jacobi polynomials. Madhekar and Thakare obtained recurrence relations, operational formulae, generating functions, bi-orthogonality, multilinear and multilateral generating function involving bi-orthogonal polynomials suggested by Jacobi polynomials. Dhanorkar and Kavthekar [3] worked on biorthogonal polynomials for the weight function $\frac{|x|^{2\mu}}{(-x^2q^2;q^2)_\infty}$. In the present paper we obtained some interesting results with some particular cases for the first set $J_n(\alpha, \beta, k: x)$. In which generating function is obtained from hypergeometric function and Manocha [8]. Also find recurrence relation from Rainville [10].

Keywords and phrases:

Generating Function Biorthogonal polynomials, Recurrence Relations, generalized hypergeometric function.

2020 Mathematics subject classification: 33C45, 33E50, 33C67, 33C20

1 Introduction:

Madhkkar and Thakare [7] constructed following pair of polynomials

$$J_n(\alpha, \beta, k; x) = \frac{(1+\alpha)k_n}{n!} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{(1+\alpha+\beta+n)_{k_j}}{(1+\alpha)_{k_j}} \left(\frac{1-x}{2}\right)^{k_j} \tag{1.1}$$

and

$$k_n(\alpha, \beta, k; x) = \sum_{r=0}^{n} \sum_{s=0}^{r} (-1)^{r+s} \binom{r}{s} \frac{(1+\beta)_n}{n!r!(1+\beta)_{n-r}} \left(\frac{s+\alpha+1}{k}\right)_n \left(\frac{x-1}{2}\right)^r \left(\frac{x+1}{2}\right)^{n-r} \tag{1.2}$$

They showed that first set $\{J_n(\alpha, \beta, k; x)\}$ is bi-orthogonal to the second set $\{k_n(\alpha, \beta, k; x)\}$ with respect to the weight function $(1-x)^\alpha(1+x)^\beta$ over the interval $(-1,1)$ where $\alpha, \beta > -1$ and k is a positive integer. For $k = 1$ both $J_n(\alpha, \beta, k; x)$ and $k_n(\alpha, \beta, k; x)$ reduces to the Jacobi polynomials $P_n^{\alpha,\beta}(x)$

Madhekar and Thakare [5, 6, 7] and Thakare and Madhekar [13, 14] obtained biorthogonality, operational formulae and generating functions of these polynomials.

The polynomials $J_n(\alpha,\beta,k;x)$ and $k_n(\alpha,\beta,k;x)$ are related to $Z_n^{\alpha}(x;k)$ and $Y_n^{\alpha}(x,k)$ respectively by the following relations

$$\lim_{\beta\to\infty} J_n\left(\alpha,\beta,k;1-\frac{2x}{\beta}\right)=Z_n^{\alpha}(x;k) \quad (1.3)$$

$$\lim_{\beta\to\infty} k_n\left(\alpha,\beta,k;1-\frac{2x}{\beta}\right)=Y_n^{\alpha}(x;k) \quad (1.4)$$

where $Z_n^{\alpha}(x;k)$ and $Y_n^{\alpha}(x;k)$ are the konhauser biorthogonal polynomials [4].

In the present paper we obtained some more interesting results for the first set $\{J_n(\alpha,\beta,k;x)\}$ and some particular cases are also noted.

2 Generating Functions

The generalized hypergeometric function is defined by

$$pF_{q[]} \begin{matrix}\alpha_1,\alpha_2,\dots,\alpha_p\\ \beta_1,\beta_2,\dots,\beta_q\end{matrix} Z = \sum\text{undOvr}n =$$
$0\infty(\alpha 1)n(\alpha 2)n \dots (\alpha p)n(\beta 1)n(\beta 1)n \dots (\beta q)nZnn!$(2.1)

Here p and q are positive integers or zero. $\alpha_1,\alpha_2,\dots,\alpha_p$ are numerator parameters and $\beta_1,\beta_2,\dots,\beta_q$ are denominator parmeters. They may be real or complex and $\beta_j\neq 0,-1,-2,\dots;\ j=1,2,\dots,9$ and Z is variable.

From definition (1.1) we easily observed that the polynomial $J_n(\alpha,\beta,k;x)$ has following hypergeometric representation.

$$J_n(\alpha,\beta,k;x)=\frac{(1+\alpha)kn}{n!}\ {}_{k+1}F_k\left[\begin{matrix}-n;\Delta(k,1+\alpha+\beta+n);\\ \Delta(k,1+\alpha);\end{matrix}\left(\frac{1-x}{2}\right)^k\right] \quad (2.2)$$

where $\Delta(m,\delta)$ stands for the sequence of m parameters

$$\frac{\delta}{m},\frac{\delta+1}{m},\dots,\frac{\delta+m-1}{m},\ m\geq 1$$

Chaundy [[1], p-62, equation (2.5)] gave the generating relation

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\ {}_{p+1}F_q\left[\begin{matrix}-n;\alpha_1,\dots,\alpha_p;\\ \beta_1,\dots,\beta_q;\end{matrix}x\right]t^n=$$
$$(1-t)^{-\lambda}\ {}_{p+1}F_q\left[\begin{matrix}\lambda;\alpha_1,\dots,\alpha_p\\ \beta_1,\dots,\beta_q\end{matrix}\ \frac{xt}{(t-1)}\right]$$
$$where|t|\leq 1 \quad (2.3)$$

Specialising the parameters in (2.3) in view of hypergeometric representation (2.2) for $J_n(\alpha,\beta,k;x)$ we get the generating relation.

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{(1+\alpha)_{kn}}J_n(\alpha,\beta-n,k;x)t^n=$$
$$(1-t)^{-\lambda}\ {}_{k+1}F_k\left[\begin{matrix}\lambda;\Delta(k,1+\alpha+\beta);\\ \Delta(k,1+\alpha);\end{matrix}\quad 2^{-k}(1-x)^k\frac{t}{(t-1)}\right]$$
$$(|t|<1) \quad (2.4)$$

Replacing x by $1-\frac{2x}{\beta}$ and using the relation (1.3) we obtain the generating function for $Z_n^{\alpha}(x;k)$ obtained by Srivastava [[10], p-245, equation 3.19].

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{(\alpha+1)_{kn}} Z_n^{\alpha}(x;k)t^n = (1-t)^{-\lambda} {}_1F_k \left[\begin{matrix}\lambda; \\ \Delta(k,\alpha+1);\end{matrix} \frac{x^k t}{(t-1)k^k}\right] \quad (2.5)$$

see also Srivastava and Manocha [[12], p-198, problem 66].

Replacing t by t/λ in the formula (2.4) and taking $\lambda \to \infty$ we get the following generating function obtained by Madhekar and Thakare [[5], p-421, equation (14)]

$$\sum_{n=0}^{\infty} \frac{J_n(\alpha,\beta-n,k;x)}{(1+\alpha)_{kn}} t^n = e^t {}_kF_k \left[\begin{matrix}\Delta(k,1+\alpha+\beta); \\ \Delta(k,1+\alpha);\end{matrix} -t\left(\frac{1-x}{2}\right)^k\right] \quad (2.6)$$

Replacing x by $1-2x$ and putting $k=1$ in (2.6) we get known generating function of Jacobi polynomials due to Srivastava and Joshi [[11], p-22, equation (4.6)]

$$\sum_{n=0}^{\infty} \frac{p_n^{(\alpha,\beta-n)}(1-2x).t^n}{(1+\alpha)_n} = e^t {}_1F_1 \left[\begin{matrix}1+\alpha+\beta; \\ 1+\alpha;\end{matrix} -xt\right] \quad (2.7)$$

putting $k=1$ in (2.6) and using Kummer's transformation.

$${}_1F_1(a;c;z) = e^z {}_1F_1(c-a,c,-z)$$

we get Feldheim's formula [[2], p-120, equation(12)]

$$\sum_{n=0}^{\infty} \frac{p_n^{(\alpha,\beta-n)}(x)}{(1+\alpha)_n} t^n = e^{(x+1)t/2} {}_1F_1 \left[\begin{matrix}-\beta; \\ 1+\alpha;\end{matrix} \frac{t(1-x)}{2}\right] \quad (2.8)$$

see also Shrivastava and Manocha [[12], p-170, problem (19)] Now reversing the orders of summation we get

$$J_n(\alpha,\beta,k;x) = \frac{(1+\alpha)_{kn}}{n!}(-1)^n\left(\frac{1-x}{2}\right)^{kn} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{(1+\alpha+\beta+n)_{kn-kj}}{(1+\alpha)_{kn-kj}} \left(\frac{1-x}{2}\right)^{kj} \quad (2.9)$$

Now replacing α by $\alpha-kn$, β by $\beta-n$ in (2.9) and using the result due to Rainville [[9], p-32, problme (8)]

$$(\alpha)_{n-k} = \frac{(-1)^k(\alpha)_n}{(1-\alpha-n)_k}$$

we get

$$J_n(\alpha-kn,\beta-n,k;x) = (-\alpha-\beta)_{kn}(-1)^n\left(\frac{x-1}{2}\right)^{kn} {}_{k+1}F_k \left[\begin{matrix}-n,\Delta(k,-\alpha); \\ \Delta(k,-\alpha-\beta);\end{matrix} (2/1-x)^k\right] \quad (2.10)$$

Specialising (2.3) in view of (2.10) we obtain the following generating function for the polynomials $J_n(\alpha,\beta,k;x)$

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n J_n(\alpha-kn,\beta-n,k;x)t^n}{(-\alpha-\beta)_{kn}} = \left[1+t\left(\frac{x-1}{2}\right)^k\right]^{-\lambda} k+1F_k \left[\begin{matrix}\lambda,\Delta(k,-\alpha); \\ \Delta(k,-\alpha-\beta);\end{matrix} \frac{(-1)^k t}{1+t\left(\frac{x-1}{2}\right)^k}\right] \quad (2.11)$$

For $k = 1$ the equation (2.11) reduces to the generating function for Jacobi polynomials $p_n^{\alpha,\beta}(x)$ given by Manocha [[8], equation (1.4)]

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n p_n^{\alpha-n,\beta-n}(x)t^n}{(-\alpha-\beta)_{kn}} = \left[1+\left(\frac{x-1}{2}\right)t\right]^{-\lambda} \quad 2F_1\left[\begin{matrix}\lambda,-\alpha; \\ -\alpha-beta;\end{matrix}\frac{-t}{1+\left(\frac{x-1}{2}\right)t}\right] \quad (2.12)$$

Replace t by t/λ and taking limit $\lambda \to \infty$ in equation (2.11) we get

$$\sum_{n=0}^{\infty} \frac{J_n(\alpha-kn,\beta-n,k;x)}{(-\alpha-\beta)_{kn}} t^n = \exp\left[-t\left(\frac{x-1}{2}\right)^k\right] \quad {}_kF_k\left[\begin{matrix}\Delta(k,-\alpha); \\ \Delta(k,-\alpha-\beta);\end{matrix}(-1)^k t\right] \quad (2.13)$$

putting $k = 1$ the equation (2.13) reduces to the following generating function

$$\sum_{n=0}^{\infty} \frac{1}{(-\alpha-\beta)_n} p_n^{\alpha-n,\beta-n}(x)t^n = \exp\left[-t\left(\frac{x-1}{2}\right)\right] \quad {}_1F_1\left[\begin{matrix}-\alpha; \\ -\alpha-\beta;\end{matrix}-t\right] \quad (2.14)$$

Replace t by $(-t)$, α by β and β by α in equation (2.14) we obtain

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(-\alpha-\beta)_n} p_n^{\beta-n,\alpha-n}(x)t^n = \exp\left[t\left(\frac{x-1}{2}\right)\right]_1 F_1\left[\begin{matrix}-\beta; \\ -\alpha-\beta;\end{matrix}t\right] \quad (2.15)$$

Now using the result from Rainville [[10], p-256]

$$p_n^{\alpha,\beta}(-x) = (-1)^n p_n^{\beta,\alpha}(x)$$

we obtain

$$\sum_{n=0}^{\infty} \frac{p_n^{\alpha-n,\beta-n}(-x)t^n}{(-\alpha-\beta)_n} = \exp\left[t\left(\frac{x-1}{2}\right)\right]_1 F_1\left[\begin{matrix}-\beta; \\ -\alpha-\beta;\end{matrix}t\right] \quad (2.16)$$

Now replcing x by $-x$ in the equation (2.16) we get

$$\sum_{n=0}^{\infty} \frac{p_n^{\alpha-n,\beta n}(x)}{(-\alpha-\beta)_n} t^n = \exp\left[-t\left(\frac{x+1}{2}\right)\right]_1 F_1\left[\begin{matrix}-\beta; \\ -\alpha-\beta;\end{matrix}t\right] \quad (2.17)$$

The generating function (2.17) is due to Manocha [8].

3. Recurrence Relation

Consider the following formula from Rainville [[9], p-107, problem 12]

$$\frac{d}{dz_p} F_q\left[\begin{matrix}a_1,\dots,a_p; \\ b_1,\dots,b_q;\end{matrix}z\right] = \frac{\prod_{m=1}^{p} a_m}{\prod_{j=1}^{q} b_j}\ {}_p F_q\left[\begin{matrix}a_1+1,\dots,a_p+1; \\ b_1+1,\dots,b_q+1;\end{matrix}z\right] \quad (3.1)$$

From (2.2) we have

$$\frac{d}{dx} J_n(\alpha,\beta,k;x) = \frac{d}{dx}\frac{(1+\alpha)_{kn}}{n!}\ {}_{k+1}F_k\left[\begin{matrix}-n;\Delta(k,1+\alpha+\beta+n); \\ \Delta(k,1+\alpha);\end{matrix}\left(\frac{1-x}{2}\right)^k\right]$$

using (3.1) we get

$$\frac{d}{dz} J_n(\alpha,\beta,k;x) = \frac{(1+\alpha)_{kn}}{(n-1)!} k2^{-k}(1-x)^{k-1}\frac{(1+\alpha+\beta+n)}{(1+\alpha)_k}$$

$${}_{k+1}F_k\left[\begin{matrix}-n+1,\Delta(k,1+\alpha+\beta+k+n); \\ \Delta(k,1+\alpha+k);\end{matrix}\left(\frac{1-x}{2}\right)^k\right]$$

In view of hypergeometric representation of $J_n(\alpha,\beta,k;x)$ we obtain the following recurrence relation

$$D[J_n(\alpha,\beta,k;x)] = k.2^{-k}(1-x)^{k-1}(1+\alpha+\beta+n)_k J_{n-1}(\alpha+k,\beta+1,k;x) \quad (3.2)$$

where $D=\frac{d}{dx}$ The equation (3.2) can be written as

$$(1-x)^{1-k}D[J_n(\alpha,\beta,k;x)] = k.2^{-k}(1+\alpha+\beta+n)_k J_{n-1}(\alpha+k,\beta+1,k;x)$$

It is not difficult to observe that

$$[(1-x)^{1-k}D]^m J_n(\alpha,\beta,k;x) = k^m 2^{-mk} \prod_{i=0}^{m-1} (1+\alpha+\beta+n+ki)_k J_{n-m}(\alpha+mk,\beta+m,k;x) \quad (3.3)$$

where $n \geq m > 0$

For $k=1$ the equation (3.3) reduces to the result for Jacobi polynomials

$$D^m p_n^{(\alpha,\beta)}(x) = 2^{-m}(1+\alpha+\beta+n)_m p_{n-m}^{(\alpha+m,\beta+m)}(x) \quad (3.4)$$

see Rainville [[9], p-263, equation (3)]

Madhekar and Thakare [[5], p-421, equation (16) & (17)] obtained following recurrence relation for $J_n(\alpha,\beta,k;x)$

$$(x-1)DJ_n(\alpha,\beta,k;x) = nkJ_n(\alpha,\beta,k;x) - k(kn-k+\alpha+1)_k J_{n-1}(\alpha,\beta+1,k;x) \quad (3.5)$$

$$(x-1)DJ_n(\alpha,\beta,k;x) = (kn+\alpha)J_n(\alpha-1,\beta+1,k;x) - \alpha J_n(\alpha,\beta,k;x) \quad (3.6)$$

Eliminating $(x-1)DJ_n(\alpha,\beta,k;x)$ from (3.5) and (3.6) we can easily obtain

$$J_n(\alpha,\beta,k;x) = k(1+\alpha+nk-k)_{k-1}J_{n-1}(\alpha,\beta+1,k,x) + J_n(\alpha-1,\beta+1,k;x) \quad (3.7)$$

Now using (3.2) in (3.5) we have

$$-(1-x)^k 2^{-k}(1+\alpha+\beta+n)_k J_{n-1}(\alpha+k,\beta+1,k;x) = nJ_n(\alpha,\beta,k;x) - (kn-k+\alpha+1)_k J_{n-1}(\alpha,\beta+1,k;x)$$

Replacing n by $(n+1)$ and β by $(\beta-1)$ in above equation we get the following recurrence relation

$$(1-x)^k 2^{-k}(1+\alpha+\beta+n)_k J_n(\alpha+k,\beta,k;x) = (1+\alpha+kn)_k J_n^k(\alpha,\beta,k;x) - (n+1)J_{n+1}(\alpha,\beta-1,k;x) \quad (3.8)$$

Similarly using (3.2) in (3.6) we get another recurrence relation for $J_n(\alpha,\beta,k;x)$

$$k.2^{-k}(1-x)^k(1+\alpha+\beta+n)_k J_{n-1}(\alpha+k,\beta+1,k;x) = \alpha J_n(\alpha,\beta,k;x) - (kn+\alpha)J_n(\alpha-1,\beta+1,k;x) \quad (3.9)$$

References

[1] T. W. Chaundy : An extension of hypergeometric functions, *Quart, Jour. Math, Oxford Ser*, 14 (1943) 55–78.

[2] E. Feldheim : Relations entre les polynomes de Jacobi Laguerre et Hermite, *Acta Math*, 74, (1941), 117–138.

[3] N. M. Kavathekar, Gajanan A. Dhanorkar, T.B.Jagtap: Biorthogonal polynomials for the weight function.., Eur. Chem. Bull. 2023, 12 (S3), 6857 – 6865. DOI: 10.31838/ecb/2023.12.s3.768

[4] J. D. E Konhauser : Bi-orthogonal polynomials suggested by the Laguerre polynomials, *Jour, Math*, 21 (1967), 303–314.

[5] H. C. Madhekar and N. K. Thakare : Biorthogonal polynomials suggested by the Jacobi polynomials, *Jour, Math*, 100, (1982), 417–424.

[6] H. C. Madhekar and N. k. Thakare : On obtaining multilateral generating functions, *Indian Journal Pure Appl. Math.*, 15(2), (1984), 149–160.

[7] H. C. Madhekar and N. k. Thakare : Multilateral generating functions for bi-orthogonal polynomials suggested by Jacobi polynomials, *(*Indian Jour. Pure and Appl. Math.), 15(2), (1984), 161–170.

[8] H. L. Manocha : Some bilinear generating functions for Jacobi polynomials, *Proc. Camb. Phil. Soc.*, 63, (1967)m 457–459.

[9] Earl D. Rainville : *Special Functions*, The Macmillan Company, New York, (1960). Reprinted by Chelsea Publ. Co. Bronx, New York, (1971).

[10] H. M. Srivastava : Some biorthogonal polynomials suggested by the Laguerre polynomials, *Pacific Jour. Math.*, 98, (1982), 235–250.

[11] H. M. Srivastava and C. M. Joshi : Certain double Whittaker transforms of generalised hypergeometric functions, *Yokahama Math. Jour.*, 15, (1967), 17–32.

[12] H. M. Srivastava and H. L. Manocha : *A Treatise on Generating Functions*, Ellis Harwood Ltd, (1984).

[13] N. K. Thakare and H. C. Madhekar : On the second set $\{k_n(a, b, k; x)\}$ of Biorthogonal polynomials suggested by Jacobi polynomials, *Bulletin of the Institute of Mathematics Academia Sinica*, 12(4), (1984), 413–422.

[14] N. K. Thakare and H. C. Madhekar : Results involving a pair of biorthogonal polynomials related to Jacobi polynomial, *Bull. Inst. Math. Academia Sinica*, 14(3), (1986), 251–265.